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REDUCED CONVEX BODIES IN THE PLANE

BY

MAREK LASSAK

Instytut Matematyki i Fizyki ATR, Bydgoszcz, Poland

ABSTRACT

A convex body R of Euclidean d-space E^d is called reduced if there is no convex body properly contained in R of thickness equal to the thickness $\Delta(R)$ of R. The paper presents basic properties of reduced bodies in E^2. Particularly, it is shown that the diameter of a reduced body $R \subset E^2$ is not greater than $\sqrt{2}\Delta(R)$, and that the perimeter is at most $(2 + \frac{1}{2}\pi)\Delta(R)$. Both the estimates are the best possible.

1. Introduction

By a *convex body* of Euclidean d-space E^d we understand a compact convex set with nonempty interior. The set of all points between two parallel hyperplanes, including the hyperplanes, is called a *strip*. Let C be a convex body of E^d. The symbol $\mathrm{bd}(C)$ denotes the boundary of C. By a *C-strip* we mean a strip containing C whose both bounding hyperplanes support C. If a point $p \in \mathrm{bd}(C)$ belongs to a hyperplane bounding a C-strip S, we say that S *passes through* p. The C-strip whose bounding hyperplanes are perpendicular to a direction ℓ is denoted by $S(C, \ell)$. By the *width* $w(C, \ell)$ *of C in a direction* ℓ we understand the distance of the hyperplanes bounding $S(C, \ell)$. The number $\Delta(C) = \min w(C, \ell)$, where the minimum is taken over all directions ℓ, is called the *thickness* of C. The diameter of C is denoted by $\mathrm{diam}(C)$ and the perimeter of $C \subset E^2$ by $\mathrm{perim}(C)$.

A convex body $R \subset E^d$ is called *reduced* if $\Delta(P) < \Delta(R)$ for every convex body $P \subset R$ different from R. This notion was introduced by Heil [6] who asked if every reduced strictly convex body of E^d is of constant width. Dekster [1] gave the positive answer for $d = 2$. He also proved [2] that every reduced

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almost spherically convex body of E^d is of constant width. We present further properties of reduced bodies.

The importance of the notion of reduced body follows, in particular, from the fact that for many extremal problems concerning the thickness of convex bodies it is sufficient to limit the considerations to the class of reduced bodies. First of all, we mean the classical problems of finding or estimating the minimum of the ratio of the i-th volume to the thickness of a convex body. Other problems of this kind are how large can be a regular simplex and a cube contained in every convex body $C \subset E^d$ of thickness 1. Also for most problems discussed in [4] it is sufficient to consider only reduced bodies.

In order to make the subject closer to the reader, let us present a few examples of reduced bodies. It is clear that all convex bodies of constant width are reduced. Another example is the set

$$\{(x_1, \ldots, x_d);\, x_1^2 + \cdots + x_d^2 \leqq r^2 \quad \text{and} \quad x_i \geqq 0 \qquad \text{for } i = 1, \ldots, d\},$$

whose 2-dimensional case, called the *quarter of a disk*, plays an important part in this paper. The convex hull of points whose polar coordinates φ, r fulfil $|\varphi| \leqq \frac{1}{3}\pi$ and $r \in \{-\frac{1}{3}, \frac{2}{3}\}$ is a reduced body. All regular odd-gons are reduced. There are non-regular reduced polygons, for example, the pentagon with vertices $w_1(0, 0)$, $w_2(\lambda, 0)$, $w_3(1, \sqrt{2} - 1)$, $w_4(\sqrt{2} - 1, 1)$ and $w_5(0, \lambda)$, where $\lambda = 0.509\ldots$ is the number such that w_2 is in the distance 1 from the straight line through w_4 and w_5.

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2. A theorem on extreme points of reduced bodies

Exceptionally, this short section is formulated for reduced bodies in E^d. The considerations do not differ from those when $d = 2$.

THEOREM 1. *Through every extreme point of a reduced body $R \subset E^d$ an R-strip of thickness $\Delta(R)$ passes.*

PROOF. Consider an extreme point e of R. Denote by B_i the open ball of radius $\Delta(R)/i$ centered at e, where $i = 2, 3, \ldots$. The convex hull R_i of $R \setminus B_i$ is a convex body for $i = 2, 3, \ldots$. Since e is an extreme point of R, then R_i is a proper subset of R for $i = 2, 3, \ldots$. Since R is reduced, $\Delta(R_i) < \Delta(R)$ for $i = 2, 3, \ldots$. Thus an R_i-strip S_i of thickness $\Delta(R_i)$ passes through a point $p_i \in \mathrm{bd}(R_i)$ contained in the closure of B_i, $i = 2, 3, \ldots$. Since the sequence

$\{p_i\}$ tends to e and since $\bigcup_{i=2}^{\infty} R_i = R \setminus \{e\}$, there is a subsequence of the sequence $\{S_i\}$ convergent to an R-strip S passing through e. From $\Delta(S_i) = \Delta(R_i) < \Delta(R)$ for $i = 2, 3, \ldots$ we obtain that $\Delta(S) \leqq \Delta(R)$. Since $R \subset S$, we have $\Delta(S) = \Delta(R)$, which ends the proof.

If through every boundary point of a convex body exactly one supporting hyperplane passes, the body is called *smooth*. From Theorem 1 and from the obvious fact that every supporting hyperplane of a convex body passes through an extreme point of this body we obtain

COROLLARY 1 ([5]). *Every smooth reduced convex body in E^d is of constant width.*

3. Basic properties of reduced bodies in E^2

If the oriented positive angle from a direction ℓ_1 to a different direction ℓ_2 in E^2 is smaller than π, we write $\ell_1 \prec \ell_2$. By $\ell_1 \preccurlyeq \ell_2$ we mean $\ell_1 \prec \ell_2$ or $\ell_1 = \ell_2$. If $\ell_1 \preccurlyeq \ell_2$ and $\ell_1 \preccurlyeq \ell \preccurlyeq \ell_2$, then we say that ℓ is *between* ℓ_1 and ℓ_2. If $\ell_1 \prec \ell_2$ and $\ell_1 \prec \ell \prec \ell_2$, then ℓ is said to be *strictly between* ℓ_1 and ℓ_2. The closed segment connecting points x and y is denoted by xy and the vector with the origin x and the endpoint y by $\overrightarrow{xy}$. The symbol $|xy|$ stands for the distance of x and y. If $x, y \in \mathrm{bd}(C)$, by $\widehat{xy}$ we denote the arc of $\mathrm{bd}(C)$ from x, in the positive orientation, to y. By $|\widehat{xy}|$ we mean the length of $\widehat{xy}$.

THEOREM 2. *Let $R \subset E^2$ be a reduced body. For every direction ℓ such that $w(R\ \ell) = \Delta(R)$ there are unique points $a(\ell)$ and $b(\ell)$ in* $\mathrm{bd}(R)$ *such that $|a(\ell)b(\ell)| = \Delta(R)$ and that $\overrightarrow{a(\ell)b(\ell)}$ is of the direction ℓ.*

The notation $a(\ell)$, $b(\ell)$ introduced in Theorem 2 is used in the whole paper. The author conjectures that Theorem 2 holds true in E^d.

THEOREM 3. *Let m_1 and m_2, where $m_1 \prec m_2$, be directions such that $w(R, m_1) = \Delta(R) = w(R, m_2)$ and that $w(R, m) > \Delta(R)$ for every direction m strictly between m_1 and m_2. Then the segments $a(m_1)a(m_2)$ and $b(m_1)b(m_2)$ are in* $\mathrm{bd}(R)$. *One of them is perpendicular to m_1 and the another to m_2. Moreover, the angle ψ from m_1 to m_2 is not greater than $\frac{1}{2}\pi$ and $|a(m_1)a(m_2)| = \Delta(R)\tan\frac{1}{2}\psi = |b(m_1)b(m_2)|$.*

PROOF OF THEOREMS 2 AND 3. We provide the proof of both theorems together because the notation introduced in Theorem 2 enables a simple formulation of Theorem 3 and because the result of Theorem 3 is needed for the proof of Theorem 2. Our proof consists of three parts.

Part I. Since R is a convex body, for every direction ℓ such that $w(R, \ell) = \Delta(R)$ there are $a(\ell)$, $b(\ell) \in \mathrm{bd}(R)$ such that $|a(\ell)b(\ell)| = \Delta(R)$ and that $\overrightarrow{a(\ell)b(\ell)}$ is of the direction ℓ (see [3], p. 77).

Part II. Let m_1 and m_2 be directions as in the assumptions of Theorem 3. By Part I of our proof there exist consecutive (according to the positive orientation) points $a_1, a_2, b_1, b_2 \in \mathrm{bd}(R)$ such that $|a_ib_i| = \Delta(R)$ and that $\overrightarrow{a_ib_i}$ is of the direction m_i, $i = 1, 2$. Let w, x, y, z be consecutive vertices of the rhombus

$$Q = S(R, m_1) \cap S(R, m_2)$$

such that $a_1 \in wx$, $a_2 \in xy$, $b_1 \in yz$ and $b_2 \in zw$.

We will show that $\widehat{a_1a_2} = a_1a_2$. Suppose the opposite. Of course, $a_1 \neq a_2$. Let K denote this straight line parallel to a_1a_2 which supports R at a point of $\widehat{a_1a_2}$. Let L be the line parallel to K passing in the half of the distance between a_1a_2 and K. The intersection of R with the halfplane bounded by L and containing a_1a_2 is denoted by R_L.

Let c_1 denote the intersection of L with wx. Observe that there are directions ℓ_1, ℓ_2 such that $m_1 < \ell_1 < \ell_2 < m_2$ and that

$$(1) \quad w(R_L, m) \geqq \Delta(R) \qquad \text{for } m_1 \leqslant m \leqslant \ell_1 \qquad \text{and for} \qquad \ell_2 \leqslant m \leqslant m_2.$$

There is an $\varepsilon > 0$ such that

$$(2) \qquad \text{if } \ell_1 \leqslant m \leqslant \ell_2, \quad \text{then } w(R, m) \geqq \Delta(R) + \varepsilon.$$

Really, in the opposite case a sequence $\{n_i\}$ of directions between ℓ_1 and ℓ_2 would exist such that $\lim_{i\to\infty} w(R, n_i) = \Delta(R)$, and thus, by compactness arguments, a direction n between ℓ_1 and ℓ_2 would exist such that $w(R, n) = \Delta(R)$, a contradiction with the assumptions of Theorem 3.

Let M be the line parallel to L passing between K and L in the distance ε from K. The line M dissects R into two closed subsets; this containing R_L is denoted by R_M. Observe that

$$(3) \qquad w(R_M, m) \geqq w(R, m) - \varepsilon \qquad \text{for every direction } m.$$

From (1), (2), (3) and from $R_L \subset R_M \subset R$ we see that $\Delta(R_M) = \Delta(R)$ which contradicts the fact that R is a reduced body. Analogously, the assumption that $\widehat{b_1b_2}$ is not b_1b_2 also leads to a contradiction. Hence

(4) $$\widehat{a_1a_2} = a_1a_2 \quad \text{and} \quad \widehat{b_1b_2} = b_1b_2.$$

Suppose that $a_1 \neq x$ and $a_2 \neq x$. Then both the angles $\angle b_1a_1a_2$ and $\angle b_2a_2a_1$ are smaller than $\frac{1}{2}\pi$. In virtue of (4) we obtain that the width of R in a direction between m_1 and m_2 is smaller that $\Delta(R)$. A contradiction. Thus a_1 or a_2 coincides with x. Similarly we show that b_1 or b_2 coincides with z. Since Q is a rhombus,

(5) either $a_1 = x$ and $b_2 = z$, or $a_2 = x$ and $b_1 = z$.

Consequently,

(6) one of each two opposite sides of Q contains only one point of R.

From (4) and (5) we see that one of the segments a_1a_2, b_1b_2 is perpendicular to m_1 and the other to m_2. The assumption that R is reduced and (4) show that the angle ψ from m_1 to m_2 is not greater than $\frac{1}{2}\pi$.

From (5) it results that $|a_1a_2| = |b_1b_2|$. Consequently, the angles between the diagonals and the bases of the trapezium $a_1a_2b_1b_2$ are $\frac{1}{2}\psi$. Moreover, the diagonals are perpendicular to the arms of the trapezium. From $|a_1b_1| = \Delta(R) = |a_2b_2|$ we obtain $|a_1a_2| = \Delta(R)\tan\frac{1}{2}\psi = |b_1b_2|$.

Part III. We show the uniqueness of $a(\ell)$ and $b(\ell)$ in Theorem 2.

Suppose the opposite: there are $a'(\ell) \neq a(\ell)$ and $b'(\ell) \neq b(\ell)$ in bd(R) such that $\overrightarrow{a'(\ell)b'(\ell)} = \overrightarrow{a(\ell)b(\ell)}$. Since the parallel segments $a(\ell)a'(\ell)$ and $b(\ell)b'(\ell)$ are in bd(R), there is a direction $\ell_0 < \ell$ such that $w(R, \ell_0) = \Delta(R) = w(R, \ell)$ and that $w(R, n) > \Delta(R)$ for every direction n strictly between ℓ_0 and ℓ. Consequently Part II of our proof can be applied to directions ℓ_0 and ℓ. Particularly, (6) contradicts $a'(\ell) \neq a(\ell)$ and $b'(\ell) \neq b(\ell)$. Thus $a(\ell)$ and $b(\ell)$ are unique.

Theorems 2 and 3 are proved.

Since $w(R, \ell)$ is a continuous function of ℓ, then $w(R, m) > \Delta(R)$ implies that there are m_1, m_2 such that $m_1 < m < m_2$ and that $w(R, \ell) > \Delta(R)$ for every ℓ between m_1 and m_2. Hence Theorem 3 and the fact that bodies of constant width do not have segments in the boundaries imply

Corollary 2 ([1]). *Every strictly convex reduced body in E^2 is of constant width.*

Corollary 3. *Let $w(R, \ell_1) = \Delta(R) = w(R, \ell_2)$ and let $a(\ell_1) = a(\ell_2)$. Then $w(R, \ell) = \Delta(R)$ for every direction ℓ between ℓ_1 and ℓ_2. Moreover, the* *shorter arc of the circle with the centre* $a(\ell_1) = a(\ell_2)$ *and radius* $\Delta(R)$ *connecting* $b(\ell_1)$ *and* $b(\ell_2)$ *is in* $\mathrm{bd}(R)$.

PROOF. If $w(R, k) > \Delta(R)$ for a direction k strictly between ℓ_1 and ℓ_2, then by the continuity of $w(R, \ell)$ there are directions k_1, k_2 such that $w(R, m) > \Delta(R)$ for every m with $k_1 < m < k_2$. From Theorem 3 we see that $a(k_1) \neq a(k_2)$. This contradicts the obvious fact that $a(k_1)$ and $a(k_2)$ coincide with $a(\ell_1) = a(\ell_2)$. Hence $w(R, \ell) = \Delta(R)$ for every ℓ between ℓ_1 and ℓ_2. The second statement of Corollary 3 is obvious now.

THEOREM 4. *Let the intersection of a reduced body R with a supporting line L of R be a segment* x_1x_2, *where* $x_1 \neq x_2$.

(a) *We have* $w(R, \ell) = \Delta(R)$ *for* ℓ *perpendicular to* x_1x_2.

(b) *The parallel to L opposite supporting line L′ of R supports R at exactly one point y and the projection x of y on L belongs to* x_1x_2.

(c) *Let* $i \in \{1, 2\}$ *and* $x \neq x_i$. *Let* $y_i \notin L'$ *fulfil* $|x_iy_i| = \Delta(R)$ *and* $|yy_i| = |xx_i|$. *Then* $yy_i \subset \mathrm{bd}(R)$.

PROOF. (a) Let ℓ be one of the two directions perpendicular to x_1x_2. Suppose that $w(R, \ell) > \Delta(R)$. Then there are directions k_1, k_2 such that $k_1 < \ell < k_2$, $w(R, k_1) = \Delta(R) = w(R, k_2)$ and that $w(R, k) > \Delta(R)$ for every direction k strictly between k_1 and k_2. This, Theorem 1, and the fact that x_1, x_2 are extreme points of R imply that the strip $S(R, k_i)$ passes through x_i, $i = 1, 2$, the lines through x_i bounding $S(R, k_i)$, where $i = 1, 2$, intersect at a point x_0. Since $x_0 \notin x_1x_2$, we have $x_0 \notin R$. This contradicts (5) (x_0 plays the part of x or z there).

(b) Let m_1 be one of the two directions perpendicular to L. By (a) we have $w(R, m_1) = \Delta(R)$. Suppose that $L' \cap R$ consists of more than one point. Applying Theorem 2 we see that one of the two segments $L \cap R$, $L' \cap R$ has the form a_1a_0 and the other has the form b_0b_1, where $a_1 = a(m_1) \neq a_0$ and $b_1 = b(m_1) \neq b_0$. Thus there is a direction $m_2 \neq m_1$ such that $w(R, m_1) = \Delta(R) = w(R, m_2)$ and that $w(R, m) > \Delta(R)$ for every m strictly between m_1 and m_2. Hence the assumptions of Theorem 3 are fulfilled. Applying (6) we get a contradiction with the fact that R has more than one point in L as well as more than one point in L'.

(c) Consider the case $i = 1$. Let the order of x_2, x, x_1 on $\mathrm{bd}(R)$ correspond to the orientation. Denote by m_1 the direction of the vector $\overrightarrow{xy}$ and by m_0 the direction of $\overrightarrow{x_1y_1}$. Clearly, $m_1 < m_0$. There is a direction m_2 such that $m_1 < m_2$, $w(R, m_2) = \Delta(R)$ and $w(R, m) > \Delta(R)$ for every m strictly between m_1 and m_2. Since $x_1, x, y \in R$, we have $m_0 \preccurlyeq m_2$. If $m_0 \prec m_2$, then in virtue of Theorem 3 there exists $x_0 \in \mathrm{bd}(R)$ such that x_1 is an interior point of the segment x_0x. A contradiction. Thus $m_0 = m_2$. By Theorem 3 we see that $yy_1 \subset \mathrm{bd}(R)$.

The left and the right supporting lines of a convex body $C \subset E^2$ are called *extreme supporting lines of* C.

THEOREM 5. *Let $R \subset E^2$ be a reduced convex body. We have $w(R, \ell) = \Delta(R)$ if and only if at least one of the two supporting lines of R perpendicular to ℓ is an extreme supporting line of R.*

PROOF. Let a straight line L perpendicular to ℓ be an extreme supporting line of R, say left. If L contains more than one point of R, by Theorem 4 we see that $w(R, \ell) = \Delta(R)$. Consider the case when L supports R only at one point p. There is a sequence $\{p_i\}$ of different points of $\mathrm{bd}(R)$ monotonically convergent to p from the left and such that p_ip_{i+1} is not in $\mathrm{bd}(R)$ for $i = 1, 2, \ldots$. The straight line through p_i and p_{i+1}, where $i \in \{1, 2, \ldots\}$, dissects R into two closed convex subsets; the subset containing p is denoted by R_i. Since R is reduced and since R_i is properly contained in R, then there is an R_i-strip $S(R_i, \ell_i)$ of thickness smaller than $\Delta(R)$. Observe that $S(R, \ell)$ is the limit of $S(R_i, \ell_i)$ as i tends to ∞. Hence $w(R, \ell) = \Delta(R)$.

Suppose that both the supporting lines of R perpendicular to ℓ are not extreme. Each supports R at exactly one point; denote them by q and r. Since the lines are not extreme, there are directions ℓ_1 and ℓ_2 such that $\ell_1 \prec \ell \prec \ell_2$ and that for every m between ℓ_1 and ℓ_2 the R-strip $S(R, m)$ passes through q and r. Consequently, $w(R, \ell) > \Delta(R)$.

COROLLARY 4. *Through every boundary point of a reduced body $R \subset E^2$ an R-strip of width $\Delta(R)$ passes.*

THEOREM 6. *For every extreme point e of a reduced body $R \subset E^2$ there exists a direction m such that $w(R, m) = \Delta(R)$ and $a(m) = e$.*

PROOF. Denote the left and the right supporting lines of R at e by L_1 and L_2. Let ℓ_i be the direction perpendicular to L_i and showing from e to the halfplane bounded by L_i and containing R, $i = 1, 2$.

Case 1. If L_1 or L_2 supports R only at e, apply Theorems 2 and 5.

Case 2. Let L_1, L_2 be different and let both contain nondegenerate segments of $\mathrm{bd}(R)$. If $w(R, \ell) > \Delta(R)$ for every ℓ strictly between ℓ_1 and ℓ_2, we get a contradiction with Theorems 2 and 3. Hence $w(R, m) = \Delta(R)$ for a direction m strictly between ℓ_1 and ℓ_2. Of course, $a(m) = e$.

Case 3. Let L_1 coincide with L_2 and have in common with bd(R) a segment x_1x_2, where $x_1 = e$. We apply part (c) of Theorem 4. If x taking place in Theorem 4 is different from x_1, then we get a contradiction with $L_1 = L_2$. Thus $x = x_1$ which means that $a(\ell_1) = e$.

If e is an exposed point of R, then Case 3 of the above proof is impossible. This leads us to the next corollary.

COROLLARY 5. *For every exposed point e of a reduced body $R \subset E^2$ there is an R-strip of thickness $\Delta(R)$ with a bounding line strictly supporting R at e.*

The author conjectures that many results of this section can be generalized for reduced bodies in E^d, particularly, parts (a) and (b) of Theorem 4 (where a face would play the part of the segment x_1x_2), Theorem 6, and Corollaries 4 and 5.

4. Reduced polygons

Let $V = v_1v_2\cdots v_n$ be a convex n-gon. If $k \notin \{1, \ldots, n\}$, then by v_k we understand the vertex v_m, where $m = k \pmod n$ and $m \in \{1, \ldots, n\}$.

THEOREM 7. *Every reduced polygon has an odd number of vertices. A convex odd-gon $V = v_1v_2\cdots v_n$ is reduced if and only if for every $i \in \{1, \ldots, n\}$ the projection of v_i on the line L_i through $v_{i+(n-1)/2}$ and $v_{i+(n+1)/2}$ is strictly between the two vertices and if the distance from v_i to L_i is the same for every $i \in \{1, \ldots, n\}$.*

PROOF. Let $V = v_1v_2\cdots v_n$ be a reduced n-gon. By Corollary 5 for every $i \in \{1, \ldots, n\}$ there is a direction ℓ_i such that the V-strip $S(V, \ell_i)$ has thickness $\Delta(V)$ and that a line bounding the strip strictly supports V at v_i. Denote by t_i the perpendicular projection of v_i on the opposite line bounding the strip $S(V, \ell_i)$, $i = 1, \ldots, n$. Observe that t_i is in the relative interior of a side T_i of V for $i = 1, \ldots, n$ (in the opposite case we could find a direction ℓ_i' close to ℓ_i such that $w(S, \ell_i') < \Delta(V)$). Since the segments $a(\ell_1)b(\ell_1), \ldots, a(\ell_n)b(\ell_n)$ pairwise intersect, we conclude that the sides $T_1, \ldots, T_n$ are different and consecutive. Consequently, n is odd and for $i = 1, \ldots, n$ we have

(7) $$T_i = v_{i+(n-1)/2}v_{i+(n+1)/2}.$$

Of course, the distance from v_i to L_i is $\Delta(V)$ for every $i \in \{1, \ldots, n\}$.

Finally, we show the "if" part of the second statement of our Theorem. Let $i \in \{1, \ldots, n\}$. Since the perpendicular projection of v_j on L_j is strictly between $v_{j+(n-1)/2}$ and $v_{j+(n+1)/2}$ for $j = i$, for $j = i + (n-1)/2$, and for $j = i + (n+1)/2$, then the line parallel to L_i through v_i has in common with V only v_i. Since the distance from v_i to L_i is the same for every $i \in \{1, \ldots, n\}$, then for every convex body Z being a proper subset of V we have $\Delta(Z) < \Delta(V)$. Hence V is reduced.

The proof is complete.

Now, after Theorem 7, we are ready to introduce some notation in a reduced n-gon $V = v_1v_2 \cdots v_n$, where $n \in \{3, 5, 7, \ldots\}$. The side T_i (see (7)) is called *opposite* to the vertex v_i, $i = 1, \ldots, n$. The projection of v_i on T_i is denoted by t_i, $i = 1, \ldots, n$. The segment v_it_i is called *the height of V from the vertex v_i*, $i = 1, \ldots, n$. The closed segments connecting v_i with the endpoints of the opposite side T_i are called *diagonals* from v_i, $i = 1, \ldots, n$. By Theorem 3 we get

(8) $$|v_it_{i+(n+1)/2}| = |t_iv_{i+(n+1)/2}|$$

for $i = 1, \ldots, n$. Let

(9) $$\alpha_i = \angle v_{i+1}v_it_i \quad \text{and} \quad \beta_i = \angle t_iv_iv_{i+(n+1)/2}$$

for $i = 1, \ldots, n$. By Theorem 3 for every $i \in \{1, \ldots, n\}$ we have

(10) $$\alpha_i = \angle t_{i+(n+1)/2}v_{i+(n+1)/2}v_{i+(n-1)/2},$$

(11) $$\beta_i = \angle v_iv_{i+(n+1)/2}t_{i+(n+1)/2}.$$

By (10) and (11), from the right triangle $v_it_iv_{i+(n+1)/2}$ we get $(\alpha_i + \beta_i) + \beta_i = \frac{1}{2}\pi$. Thus for $i \in \{1, \ldots, n\}$ we have

(12) $$\alpha_i = \tfrac{1}{2}\pi - 2\beta_i \quad \text{and} \quad \beta_i = \tfrac{1}{4}\pi - \tfrac{1}{2}\alpha_i.$$

From (9) and (10) we see that $2\sum_{i=1}^{n} \alpha_i = (n-2)\pi$. By (12) we get

(13) $$\sum_{i=1}^{n} \beta_i = \tfrac{1}{2}\pi.$$

THEOREM 8. *Let $V = v_1v_2 \cdots v_n$ be a reduced n-gon and let $i \in \{1, \ldots, n\}$. The angles between the height from v_i and the sides of V ending at v_i are at least*

$\frac{1}{6}\pi$. The angles between the height from v_i and the diagonals from v_i are at most $\frac{1}{6}\pi$. In each of the above properties, an angle equal to $\frac{1}{6}\pi$ implies that V is a regular triangle.

PROOF. Suppose that $\alpha_j < \frac{1}{6}\pi$ for an index $j \in \{1, \ldots, n\}$. Thanks to (12), this is equivalent to the assumption that $\beta_j > \frac{1}{6}\pi$. In order to avoid extensive indices, let us put $h = j + (n+1)/2$. The intersection of the straight lines containing $v_j v_{j+1}$ and T_j is denoted by u_j. Since $\alpha_j < \beta_j$, from the triangle $v_j u_j v_h$ we have

$$|v_j v_h| + |v_h t_j| > |v_j u_j| + |u_j t_j|.$$

This and (8) imply

$$|v_j v_h| > |t_h u_j| + |u_j t_j|. \tag{14}$$

On the other hand, from (8) we obtain that

$$|\widehat{t_h t_j}| = |\widehat{v_h v_j}| \tag{15}$$

for the arcs $\widehat{t_h t_j}$ and $\widehat{v_h v_j}$ of $\mathrm{bd}(V)$. Since $\widehat{t_h t_j}$ is contained in the triangle $t_h u_j t_j$ and since $\widehat{v_h v_j}$ is disjoint with the interior of the triangle $v_j u_j v_h$, then the convexity of V and (15) imply $|v_j v_h| \leqq |t_h u_j| + |u_j t_j|$ which contradicts (14). Thus $\alpha_j \leqq \frac{1}{6}\pi$ and $\beta_j \geqq \frac{1}{6}\pi$.

If $\alpha_j = \frac{1}{6}\pi$ (which is equivalent to $\beta_j = \frac{1}{6}\pi$), then the triangle $v_j u_j v_h$ is equilateral and, by (15), it coincides with V.

The proof is complete.

In contrast to results of this section, we do not know if there exist reduced polytopes of dimension greater than 2.

5. The diameter of a reduced body

THEOREM 9. *For every reduced body $R \subset E^2$ we have*

$$\mathrm{diam}(R)/\Delta(R) \leqq \sqrt{2},$$

and the equality holds only if R is the quarter of a disk. For every reduced polygon V the inequality

$$\mathrm{diam}(V)/\Delta(V) \leqq \tfrac{2}{3}\sqrt{3}$$

is fulfilled with equality only if V is the regular triangle.

PROOF. Since R is the convex hull of the set $E(R)$ of extreme points of R, the diameters of R and $E(R)$ are equal. Thus in order to prove the first inequality of our theorem it is sufficient to show that for every $a_1, a_2 \in E(R)$ we have $|a_1a_2| \leqq \sqrt{2}\Delta(R)$.

In virtue of Theorem 6 there exists a direction ℓ_i such that the strip $S_i = S(R, \ell_i)$ has thickness $\Delta(R)$ and passes through a_i, and that $a(\ell_i) = a_i$ for $i = 1, 2$. Let $b_i = b(\ell_i)$ for $i = 1, 2$.

If $S_1 = S_2$, then Theorem 2 implies $|a_1a_2| = 0$ or $|a_1a_2| = \Delta(R)$.

If $S_1 \neq S_2$, then the segments a_1b_1 and a_2b_2 intersect. Thus from $|a_1b_1| = \Delta(R) = |a_2b_2|$ and $a_2b_2 \subset S_1$ we get $|a_1a_2| \leqq \sqrt{2}\Delta(R)$.

From the above considerations we see that if $|a_1a_2| = \sqrt{2}\Delta(R)$, then a_1a_2 and b_1b_2 are perpendicular and $b_1 = b_2$. By Corollary 3 we conclude that R is the quarter of a disk.

Let $V = v_1v_2 \cdots v_n$ be a reduced n-gon and let $|v_jv_k| = \text{diam}(V)$ for some $j, k \in \{1, \ldots, n\}$. Of course, V is in the strip perpendicular to v_jv_k and containing v_j and v_k on the bounding lines. Theorem 7 implies $k = j + (n-1)/2$ or $k = j + (n+1)/2$. Thus from (9) and (11) we get

$$\text{diam}(V)/\Delta(V) = \max\{\sec \beta_i;\ i = 1, \ldots, n\}. \tag{16}$$

Theorem 8 and (16) imply that $\text{diam}(V)/\Delta(V) \leqq \sec \frac{1}{6}\pi = \frac{2}{3}\sqrt{3}$ with equality only if V is a regular triangle.

The above established equality (16) together with (13) leads us to

COROLLARY 6. *From amongst all reduced m-gons, where $m \leqq n$, of a fixed thickness only the regular n-gon has the minimal diameter.*

The author conjectures that for reduced bodies $R \subset E^d$, where $d \geqq 2$, the maximum of the ratio $\text{diam}(R)/\Delta(R)$ is $\sqrt{2}$, and that $\sqrt{2}$ is attained only if R is the convex hull of the centre of a sphere of radius r and of a maximal (by inclusion) subset of this sphere of diameter $\sqrt{2}r$ (two examples of such R are at the ends of Sections 1 and 6).

6. The perimeter of a reduced body

THEOREM 10. *For every reduced body $R \subset E^2$ we have*

$$\text{perim}(R)/\Delta(R) \leqq 2 + \tfrac{1}{2}\pi$$

with equality only for R being the quarter of a disk. For every reduced polygon V we have

$$\operatorname{perim}(V)/\Delta(V) \leqq 2\sqrt{3},$$

and the equality holds only if V is the regular triangle.

Proof. We start from three properties needed later in the proof. From the formula for the tangent of the sum of two angles we see that

$$\text{(17)}\qquad \begin{aligned} &\text{if } \xi_1 + \xi_2 + \cdots < \tfrac{1}{2}\pi \text{ for positive } \xi_1 > 0, \xi_2 > 0, \ldots, \\ &\text{then } \sum_{i=1}^{\infty} \tan \xi_i \leqq \tan \sum_{i=1}^{\infty} \xi_i. \end{aligned}$$

From the convexity of $\tan x$ between 0 and $\frac{1}{2}\pi$ we have:

$$\text{(18)}\qquad \begin{aligned} &\text{if } 0 \leqq \gamma_1 < \delta_1 \leqq \delta_2 < \gamma_1 < \tfrac{1}{2}\pi \quad \text{and if } \gamma_1 + \gamma_2 = \delta_1 + \delta_2, \\ &\qquad \text{then } \tan \delta_1 + \tan \delta_2 < \tan \gamma_1 + \tan \gamma_2. \end{aligned}$$

Here is the third property:

$$\text{(19)}\qquad \begin{aligned} &\text{if } 0 \leqq \omega_1 \leqq \tfrac{1}{5}\pi \quad \text{for } i = 1, 2, \ldots \text{ and if } \omega_1 + \omega_2 + \cdots = \tfrac{1}{2}\pi, \\ &\qquad \text{then } \sum_{i=1}^{\infty} \tan \omega_i \leqq 2 \tan \tfrac{1}{5}\pi + \tan \tfrac{1}{10}\pi. \end{aligned}$$

Let us show (19). There is an index h such that $\omega_1 + \cdots + \omega_h > \frac{3}{10}\pi$. We see that the number $\omega_0 = \omega_{h+1} + \omega_{h+2} + \cdots$ is less than or equal to $\frac{1}{5}\pi$. Applying (18) to $\omega_0, \ldots, \omega_h$ a finite number of times, and having in mind that $\omega_0 + \cdots + \omega_h = \frac{1}{2}\pi$ and that $0 \leqq \omega_i \leqq \frac{1}{5}\pi$ for $i = 0, \ldots, h$, we get

$$\tan \omega_0 + \cdots + \tan \omega_n \leqq 2 \tan \tfrac{1}{5}\pi + \tan \tfrac{1}{10}\pi.$$

Moreover, from (17) we get

$$\tan \omega_{h+1} + \tan \omega_{h+2} + \cdots \leqq \tan \omega_0.$$

Both the above inequalities give (19).

In the boundary of R there are at most countably many pairs of opposite segments described in Theorem 3. Denote them by W_j, Z_j, where $j \in J$. For every $j \in J$ denote by ψ_j the angle between the directions m_1 and m_2 (as in Theorem 3) perpendicular to W_j, Z_j, respectively. From Theorem 3 we obtain

$$\text{(20)}\qquad |W_j| = |Z_j| = (\tan \tfrac{1}{2}\psi_j)\Delta(R) \qquad \text{for } j \in J.$$

By the well-known Cauchy's formula (see [3], p. 89) we have

$$\text{perim}(R) = \int_0^\pi w(S, \ell_\alpha)\,d\alpha, \tag{21}$$

where ℓ_α is the direction such that the oriented angle from a fixed direction ℓ to ℓ_α is α. In virtue of Theorem 3, the above integral taken only over α such that $w(S, \ell_\alpha) = \Delta(R)$ is equal to $(\pi - \psi)\Delta(R)$, where $\psi = \Sigma_{j\in J}\,\psi_j$. By (20) we conclude that

$$\text{perim}(R) = \left(\pi - \psi + 2\sum_{j\in J} \tan\tfrac{1}{2}\psi_j\right)\Delta(R). \tag{22}$$

Case 1, *when we have $\psi_j \leqq \frac{2}{5}\pi$ for every $j \in J$.*

Since $x \leqq \tan x$ for $0 \leqq x < \frac{1}{6}\pi$, we see that $\pi - \psi \leqq 6\tan\frac{1}{6}(\pi - \psi)$. Thus from (22) we obtain

$$\text{perim}(R)/\Delta(R) \leqq 2\left[3\tan\tfrac{1}{6}(\pi - \psi) + \sum_{j\in J} \tan\tfrac{1}{2}\psi_j\right]. \tag{23}$$

Since J contains at most countably many indices j, and since $0 \leqq \frac{1}{2}\psi_j \leqq \frac{1}{5}\pi$ for every $j \in J$ and $0 \leqq \frac{1}{6}(\pi - \psi) \leqq \frac{1}{5}\pi$, then from (19) and from (23) we get

$$\text{perim}(R)/\Delta(R) \leqq 2[2\tan\tfrac{1}{5}\pi + \tan\tfrac{1}{10}\pi] < 2 + \tfrac{1}{2}\pi.$$

Case 2, *when there exists $k \in J$ such that $\psi_k \geqq \frac{2}{5}\pi$.*

Applying (17) to (22) we get

$$\text{perim}(R)/\Delta(R) \leqq \pi - \psi + 2\tan\tfrac{1}{2}\psi_k + 2\tan\tfrac{1}{2}(\psi - \psi_k). \tag{24}$$

Denote the endpoints of W_k by a, b, and the endpoints of Z_k by c, d such that ab, bd are perpendicular and that ac, cd are perpendicular. Let p be the intersection of the straight lines containing segments ab and cd. Of course, $\angle bac = \angle bdc = \frac{1}{2}\pi - \psi_k$. Thus

$$|bp| = |cp| = \Delta(R)\tan(\tfrac{1}{2}\pi - \psi_k) = \Delta(R)\cot\psi_k.$$

Consequently, the sum of lengths of the boundary segments of R which are in the triangle bpc is not greater than $2\Delta(R)\cot\psi_k$. Thus by (20) and from $\tan x \geqq x$ for $0 \leqq x < \frac{1}{2}\pi$ we conclude that

$$2\Delta(R)\cot\psi_k \geqq \Delta(R)\sum_{j\in K} \tan\tfrac{1}{2}\psi_j \geqq \Delta(R)\sum_{j\in K} \tfrac{1}{2}\psi_j = \tfrac{1}{2}\Delta(R)(\psi - \psi_k),$$

where $K = J \setminus \{k\}$. Consequently,

$$\psi - \psi_k \leqq 4\cot\psi_k. \tag{25}$$

It is easy to check that the function $2\tan\frac{1}{2}x - x$ is increasing for x between 0 and $\frac{1}{2}\pi$. Thus from (25) we get

$$2\tan\tfrac{1}{2}(\psi-\psi_k)-(\psi-\psi_k)\leqq 2\tan(2\cot\psi_k)-4\cot\psi_k.$$

Consequently, from (24) we obtain

$$\text{perim}(R)/\Delta(R)\leqq f(\psi_k), \tag{26}$$

where

$$f(x)=\pi-x+2\tan\tfrac{1}{2}x+2\tan(2\cot x)-4\cot x.$$

Since the second derivative

$$\frac{\tan\frac{1}{2}x}{\cos^2\frac{1}{2}x}+\frac{16\tan(2\cot x)+4\sin^2(2\cot x)\sin 2x}{\cos^2(2\cot x)\sin^4 x}$$

of $f(x)$ is positive for $\frac{2}{5}\pi\leqq x\leqq\frac{1}{2}\pi$, we see that $f(x)$ is a convex function between $\frac{2}{5}\pi$ and $\frac{1}{2}\pi$. Thus $f(\frac{1}{2}\pi)=2+\frac{1}{2}\pi$ and $f(\frac{2}{5}\pi)<2+\frac{1}{2}\pi$ imply that $f(x)\leqq 2+\frac{1}{2}\pi$ for x between $\frac{2}{5}\pi$ and $\frac{1}{2}\pi$ with equality only if $x=\frac{1}{2}\pi$. Hence (26) means that $\text{perim}(R)/\Delta(R)\leqq 2+\frac{1}{2}\pi$ for $\frac{2}{5}\pi\leqq\psi_k\leqq\frac{1}{2}\pi$ with equality only if $\psi_k=\frac{1}{2}\pi$. By Theorem 3 and Corollary 3 we have $\psi_k=\frac{1}{2}\pi$ only for R being the quarter of a disk.

Now, consider a reduced n-gon V. From (8) and (11) we see that

$$\text{perim}(V)/\Delta(V)=2\sum_{i=1}^{n}\tan\beta_i, \tag{27}$$

where β_i is defined in (9).

By Theorem 8 we have $0<\beta_i\leqq\frac{1}{6}\pi$ for $i=1,\dots,n$. Thus from (13) and (27), applying (18) a number of times, we obtain $\text{perim}(V)/\Delta(V)\leqq 6\tan\frac{1}{6}\pi=2\sqrt{3}$. From (27), (18) and from the last statement of Theorem 8 we deduce that $\text{perim}(V)/\Delta(V)=2\sqrt{3}$ only if V is a regular triangle.

Theorem 10 is proved.

By (13) and (18) we see that $\Sigma_{i=1}^{n}\tan\beta_i\geqq n\tan(\pi/2n)$ and that the equality holds only if $\beta_1=\cdots=\beta_n$. Hence from (27) we obtain

COROLLARY 7. *From amongst all reduced polygons of a fixed width and at most n vertices, only the regular n-gon has the minimal perimeter.*

From (21) and from Theorem 3 we get one more Corollary.

COROLLARY 8. *Let $R \subset E^2$ be a reduced body and let $w(R, \ell) = \Delta(R)$ for a direction ℓ. Then points $a(\ell)$ and $b(\ell)$ halve the perimeter of R.*

The author conjectures that $\text{area}(R)/\Delta(R) \leqq \frac{1}{4}\pi$ with equality only if R is a disk or the quarter of a disk. For bodies of constant width the above inequality is well known and the equality holds only if R is a disk (cf. [3], p. 128). Let us formulate a more general conjecture for the i-th volume $w_i(R)$ of R in E^d. We conjecture that for every $i \in \{0, \ldots, d-1\}$ the maximum of the ratio $w_i(R)/\Delta(R)$ over all reduced bodies $R \subset E^d$ is attained for the body

$$\{(x_1, \ldots, x_d); x_1^2 + \cdots + x_d^2 \leqq 1 \quad \text{and} \quad x_1 \geqq \sqrt{x_1^2 + \cdots + x_d^2}\}.$$

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